Math2050A Term1 2017
Tutorial 5, Oct 19
Last year's Midterm question

1(iii). Since $(f_1(x_n))_{n=1}^{\infty}$ is bounded, by Bolzano Weierstrass theorem, there is a subsequence (x_n^1) of (x_n) such that $\lim_{n\to\infty} f_1(x_n^1)$ exists. Since $(f_2(x_n^1))_{n=1}^{\infty}$ is bounded, by Bolzano Weierstrass theorem, there is a subsequence (x_n^2) of (x_n^1) such that $\lim_{n\to\infty} f_2(x_n^2)$ exists.

Repeating the process, there is $(x_n^1), (x_n^2), ...$ such that

(a) $(x_n^{k+1})_{n=1}^{\infty}$ is a subsequence of $(x_n^k)_{n=1}^{\infty}$ for any $k \in \mathbb{N}$

(b) $\lim_{n\to\infty} f_m(x_n^m)$ exists for each $m \in \mathbb{N}$

Then, we claim that (x_n^n) is a subsequence of (x_n) such that $\lim_{n\to\infty} f_m(x_n^n)$ exists for each $m = 1, 2, ...$

Tracing back, we may say that there is a strictly increasing function $g_1 : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f_1(x_{g_1(n)})$ exists. There is a subsequence (n_k) of (n) such that $\lim_{k\to\infty} f_2(x_{g_1(n_k)})$ exists. When we define $g_2 : \mathbb{N} \to \mathbb{N}$ by $g_2(k) := g_1(n_k)$, g_2 is a strictly increasing function taking values in the range of g_1. In short, there are strictly increasing functions $g_1, g_2, ... : \mathbb{N} \to \mathbb{N}$ such that

(a') Range of g_{k+1} is subset of range of g_k for any $k \in \mathbb{N}$.

(b') $\lim_{n\to\infty} f_m(x_{g_m(n)})$ exists for each $m \in \mathbb{N}$

It remains to show that $(g_n(n))_n$ is strictly increasing and $\lim_{n\to\infty} f_m(x_{g_n(n)})$ exists for each $m = 1, 2, ...$

Fix $n \in \mathbb{N}$, by (a'), $g_{n+1}(n+1) \geq g_n(n+1)$. g_n being strictly increasing implies that $g_n(n+1) > g_n(n)$. Therefore, $(g_n(n))_n$ is strictly increasing. For the second claim, fix $m \in \mathbb{N}$, by (b'), there is $l \in \mathbb{R}$ and the following holds:

for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|f_m(x_{g_m(n)}) - l| < \epsilon$ for any $n \geq N$.

Now, let $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|f_m(x_{g_m(n)}) - l| < \epsilon$ for any $n \geq N$. If $n \geq \max\{N, m\}$, note $g_n(n) = g_m(k)$ for some $k \geq n$, therefore, $|f_m(x_{g_n(n)}) - l| < \epsilon$. This shows the second claim.